Chapter 7

Einstein's Field Equations - The Main Goal of The Course

7.1 Introduction

In order to have a complete theory of gravity, we need to know

- **How particles behave in curved spacetime.**
- **How matter curves spacetime.**

The first question is answered by postulating that free particles [i.e. no force other than gravity] follow timelike or null geodesics. We will see later that this is equivalent to Newton's law $\mathbf{F} = -m\nabla\Phi$ in the week field limit [$\Phi/c^2 \ll 1$].

The second requires the analogue of $\nabla^2\Phi = 4\pi G\rho$. We first consider the vacuum case [$\rho = 0$] $\Rightarrow \nabla^2\Phi = 0$.

The easiest way to do this is to compare the geodesic deviation equation derived in the last section with its Newtonian analogue. In Newtonian theory the acceleration of two neighboring particles with position vectors $\mathbf{x}$ and $\mathbf{x} + \xi$ are:

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi(\mathbf{x})}{\partial x^i}\ , \qquad \frac{d^2(x^i + \xi^i)}{dt^2} = -\frac{\partial\Phi(\mathbf{x}+\xi)}{\partial x^i}\ , \tag{7.1}$$

so the separation evolves according to:

$$\begin{aligned} \frac{d^2\xi^i}{dt^2} &= \frac{\partial\Phi(\mathbf{x})}{\partial x^i} - \frac{\partial\Phi(\mathbf{x}+\xi)}{\partial x^i} \\ &= \frac{\partial}{\partial x^i}\left(\Phi(\mathbf{x}) - \Phi(\mathbf{x}+\xi)\right)\ . \end{aligned} \tag{7.2}$$

This gives us

$$\frac{d^2\xi^i}{dt^2} = -\frac{\partial^2\Phi}{\partial x^i \partial x^j}\xi^j \ , \tag{7.3}$$

since

$$\phi(\mathbf{x}+\xi) - \Phi(\mathbf{x}) = \xi^j \frac{\partial \phi}{\partial x^j} \ . \tag{7.4}$$

This clearly is analogous to the geodesic deviation equation

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\xi^\alpha = R^\alpha{}_{\mu\nu\beta}V^\mu V^\nu \xi^\beta \ , \tag{7.5}$$

provided we relate the quantities $-\frac{\partial^2\Phi}{\partial x^i \partial x^j}$ and $R^\alpha{}_{\mu\nu\beta}V^\mu V^\nu$

Both quantities have two free indices, although the Newtonian index runs from 1 to 3 while in the General Relativity case it runs from 0 to 3.

The Newtonian vacuum equation is $\nabla^2\Phi = 0$ which implies that

$$\tfrac{\partial^2\Phi}{\partial x^i \partial x^i} = 0 \ , \tag{7.6}$$

so we can write

$$R^\alpha{}_{\mu\nu\alpha}V^\mu V^\nu = 0 \ . \tag{7.7}$$

Since $\mathbf{V}$ is arbitrary we end up with

$$R_{\mu\nu} = 0 \ . \tag{7.8}$$

These are the vacuum Einstein field equations.

7.2 The non - vacuum field equations

The General Relativity version of $\nabla^2\Phi = 4\pi G\rho$ must contain $T^{\mu\nu}$ rather than ρ since we saw in Special Relativity that ρc^2 is just the 00 component of the energy - momentum tensor. This is expected anyway since in General Relativity all forms of energy [not just rest mass] should be a source of gravity.

To get the General Relativity version of the equations involving $T^{\mu\nu}$ we just replace the Minkowski metric $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$ and the partial derivative $(,)$ by the covariant derivative $(;)$. For example the energy - momentum tensor for a perfect fluid in curved space time is

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)U^\mu U^\nu + pg^{\mu\nu} \ , \tag{7.9}$$

and the conservation equations become

$$T^{\mu\nu}{}_{;\nu} = 0 \ . \tag{7.10}$$

In the above we have just used the strong form of the Equivalence Principle, which says that any non - gravitational law expressible in tensor notation in Special Relativity has exactly the same form in a local inertial frame of curved spacetime.

We expect the full [non - vacuum] field equations to be of the form

$$O(g) = \kappa T \ , \tag{7.11}$$

where O is a second order differential operator which is a $0/2$ tensor [since T is the stress energy tensor] and κ is a constant. The simplest operator that reduces to the vacuum field equations when $T = 0$ takes the form

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta} R \ . \tag{7.12}$$

Now since $T^{\alpha\beta}{}_{;\beta} = 0$ [$T^{\alpha\beta}{}_{,\beta} = 0$ in Special Relativity], we require $O^{\alpha\beta}{}_{;\beta} = 0$. Using $g^{\alpha\beta}{}_{;\beta} = 0$ gives

$$\left(R^{\alpha\beta} + \mu g^{\alpha\beta} R\right)_{;\beta} = 0 \ . \tag{7.13}$$

Comparing this with the double contracted Bianchi identities

$$G^{\alpha\beta}{}_{;\beta} = 0 \ , \tag{7.14}$$

we see that the constant μ has to be $\mu = -\frac{1}{2}$. Thus we are led to the field equations of General Relativity:

$$R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R = \kappa T^{\alpha\beta} \ , \tag{7.15}$$

or

$$G^{\alpha\beta} = \kappa T^{\alpha\beta} \ . \tag{7.16}$$

In general we can add a constant Λ so the field equations become

$$R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R + \Lambda g^{\alpha\beta} = \kappa T^{\alpha\beta} \ . \tag{7.17}$$

In a vacuum $T^{\alpha\beta} = 0$, so taking the trace of the field equations we get

$$R^{\alpha}{}_{\alpha} - \tfrac{1}{2} R g^{\alpha}{}_{\alpha} + \Lambda g^{\alpha}{}_{\alpha} = 0 \ . \tag{7.18}$$

Since $g^{\alpha}{}_{\alpha} = 4$ and the Ricci scalar $R = R^{\alpha}{}_{\alpha}$ we find that $R = 4\Lambda$, and substituting this back into the field equations leads to

$$R^{\alpha\beta} = \Lambda g^{\alpha\beta} \ . \tag{7.19}$$

We recover the previous vacuum equations if $\Lambda = 0$. Sometimes Λ is called the vacuum energy density.

We have ten equations [since $R^{\alpha\beta}$ is symmetric] for the ten metric components. Note that there are four degrees of freedom in choosing coordinates so only six metric components are really determinable. This corresponds to the four conditions

$$G^{\alpha\beta}{}_{;\beta} = 0 \ , \tag{7.20}$$

which reduces the effective number of equations to six.

It is very important to realize that although Einstein's field equations look very simple, they in fact correspond in general to six coupled non-linear partial differential equations.

7.3 The weak field approximation

We have to check that the appropriate limit, General Relativity leads to Newton's theory. The limit we shall use will be that of small velocities $\frac{v}{c} \ll 1$ and that time derivatives are much smaller than spatial derivatives. There are two things we must do:

- **We have to relate the geodesic equation to Newton's law of motion [i.e. the second law].**

and

- **Relate Einstein's field equations to the Newton-Poisson equation.**

Let's assume that we can find a coordinate system which is locally Minkowski [as demanded by the Equivalence Principle] and that deviations from flat spacetime are small. This means we can write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta} \ , \tag{7.21}$$

where η is small. Since we require that $g_{\delta\beta}g^{\alpha\beta} = \delta^{\alpha}{}_{\delta}$, the inverse metric is given by

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \epsilon h^{\alpha\beta} \; . \tag{7.22}$$

To work out the geodesic equations we need to work out what the components of the Christoffel symbols are:

$$\Gamma^{\gamma}{}_{\beta\mu} = \tfrac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right) \; . \tag{7.23}$$

Substituting for $g^{\alpha\beta}$ etc. in terms of $h^{\alpha\beta}$ we obtain

$$\Gamma^{\gamma}{}_{\beta\mu} = \tfrac{1}{2} \epsilon \eta^{\alpha\gamma} \left(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha} \right) \; . \tag{7.24}$$

The geodesic equations are

$$\frac{d^2 x^{\gamma}}{d\tau^2} + \Gamma^{\gamma}{}_{\beta\mu} \frac{dx^{\beta}}{d\tau} \frac{dx^{\mu}}{d\tau} = 0 \; . \tag{7.25}$$

But for a slowly moving particle $\tau \approx t$ so

$$\frac{d^2 x^{\gamma}}{dt} + \Gamma^{\gamma}{}_{\beta\mu} \frac{dx^{\beta}}{dt} \frac{dx^{\mu}}{dt} = 0 \; . \tag{7.26}$$

Also $\frac{dx^i}{dt} = O(\epsilon)$, so we can neglect terms like $\Gamma^{\gamma}{}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$. The geodesic equation reduces to

$$\frac{d^2 x^{\gamma}}{dt^2} + \Gamma^{\gamma}{}_{00} \frac{dx^0}{dt} \frac{dx^0}{dt} = 0 \; , \tag{7.27}$$

so the "space" equation (three - acceleration) is

$$\frac{d^2 x^{i}}{dt^2} + \Gamma^{i}{}_{00} \frac{dx^0}{dt} \frac{dx^0}{dt} = 0 \; . \tag{7.28}$$

Since $\frac{dx^0}{dt} = c$ we get

$$\frac{d^2 x^{i}}{dt^2} = -c^2 \Gamma^{i}{}_{00} \; . \tag{7.29}$$

Now

$$\begin{aligned} \Gamma^{i}{}_{00} &= \tfrac{1}{2} \epsilon \left(h_{i0,0} + h_{i0,0} - h_{00,i} \right) \\ &\approx -\tfrac{1}{2} \epsilon h_{00,i} \; , \end{aligned} \tag{7.30}$$

where we have neglected time derivatives over space derivatives. The spatial geodesic equation then becomes

$$\frac{d^2x^i}{dt^2} = \tfrac{c^2}{2}\epsilon h_{00,i} = \tfrac{c^2}{2}\epsilon \nabla_i h_{00} \ . \tag{7.31}$$

But Newtonian theory has

$$\frac{d^2x^i}{dt^2} = -\nabla_i \Phi \ , \tag{7.32}$$

where Φ is the gravitational potential. So we make the identification

$$g_{00} = -\left(1 + \frac{2\phi}{c^2}\right) \ . \tag{7.33}$$

This is equivalent to having spacetime with the line element

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\phi}{c^2}\right)\left(dx^2 + dy^2 + dz^2\right) \ . \tag{7.34}$$

This is what we deduced using the Equivalence Principle in section 4.5.

Let's now look at the field equations [with $\Lambda = 0$]:

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta} \ . \tag{7.35}$$

Taking the trace we get

$$\begin{aligned} R - 2R &= \kappa T^{\alpha}{}_{\alpha} = \kappa T \\ \Rightarrow R &= -\kappa T \ . \end{aligned} \tag{7.36}$$

This allows us to write the field equations as

$$R_{\alpha\beta} = \kappa \left[T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T \right] \ . \tag{7.37}$$

Let us assume that the matter takes the form of a perfect fluid, so the stress - energy tensor takes the form:

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) U_\alpha U_\beta + p g_{\alpha\beta} \ . \tag{7.38}$$

Taking the trace gives

$$T \equiv T^{\alpha}{}_{\alpha} = -c^2 \left(\rho + \frac{p}{c^2}\right) + 4p \ , \tag{7.39}$$

so the field equations become

$$R_{\alpha\beta} = \kappa \left(\rho + \frac{p}{c^2} \right) U_\alpha U_\beta + \frac{1}{2}\kappa \left(\rho - \frac{p}{c^2} \right) c^2 g_{\alpha\beta} \ . \tag{7.40}$$

The Newtonian limit is $\rho >> \frac{p}{c^2}$. This gives

$$R_{\alpha\beta} = \kappa \rho U_\alpha U_\beta + \tfrac{1}{2}\kappa \rho c^2 g_{\alpha\beta} \ . \tag{7.41}$$

Look at the 00 component of these equations:

$$\begin{aligned} R_{00} &= \kappa \rho c^2 - \tfrac{1}{2}\kappa \rho c^2 \\ &= \tfrac{1}{2}\kappa \rho c^2 \end{aligned} \tag{7.42}$$

to first order in ϵ. Now

$$R_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta,\mu} - \Gamma^\mu{}_{\alpha\mu,\beta} \tag{7.43}$$

to first order in ϵ. The $(0, 0)$ component of this equation is

$$R_{00} = \Gamma^\mu{}_{00,\mu} - \Gamma^\mu{}_{0\mu,0} \ , \tag{7.44}$$

and since spatial derivatives dominate over time derivatives, we get

$$R_{00} = \Gamma^i{}_{00,i} \ . \tag{7.45}$$

So the field equations are

$$R_{00} = \Gamma^i{}_{00,i} = -\tfrac{1}{2}\epsilon h_{00,ii} = -\tfrac{1}{2}\kappa \rho c^2 \ . \tag{7.46}$$

This is just

$$\nabla^2 \phi = \tfrac{1}{2}\kappa \rho c^4 \ . \tag{7.47}$$

Comparing this with Poisson's equation:

$$\nabla^2 \phi = 4\pi G \rho \ , \tag{7.48}$$

we see that we get the same result if the constant κ is

$$\kappa = \frac{8\pi G}{c^4} \ . \tag{7.49}$$

We can now use this result to write down the full Einstein field equations:

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = \tfrac{8\pi G}{c^4} T_{\alpha\beta} \ . \tag{7.50}$$

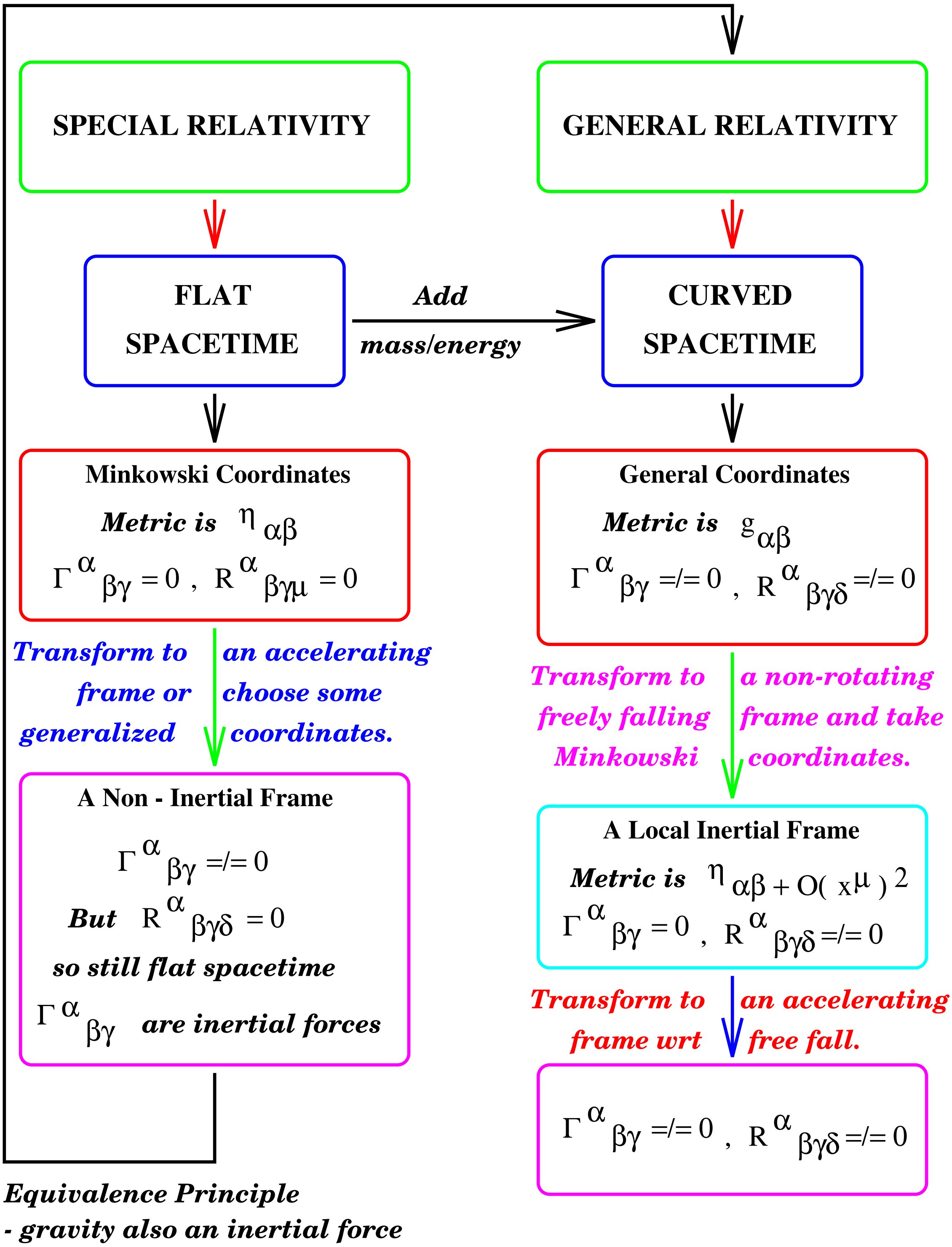

Figure 7.1: A SUMMARY OF WHAT WE HAVE DONE : –) =